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BANACH SPACES WITH TRIVIAL ISOMETRIES

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ABSTRACT

Each separable Banach space has an equivalent norm whose only isometries are $\pm$ identity. An equivalent norm on a non-separable Hilbert space is constructed so that its only isometries are $\pm$ identity.

A Banach space is said to have trivial isometries if the only isometries are $\pm$ identity. (Hence we are considering Banach spaces over the Reals.) We have two results about such Banach spaces. Theorem 1 says each separable Banach space has an equivalent norm with only trivial isometries. Theorem 2 constructs an equivalent norm on a non-separable Hilbert space with only trivial isometries.

Davis [3] has the earliest result in this area. Davis renorms l_2 so that it has only trivial isometries. His construction is more general, but it isn't known if his construction could yield Theorem 1. Davis also describes Pelcynski's $C(K)$ space with trivial isometries. Other results may be found in [1], [2].

Our construction is based on "pimples", a way of decreasing the norm so that the unit ball has two "cones" added. The Proposition in Section 1 contains most of the technical details. Although the proof seems long winded, geometrically the statment of Proposition is clear.

§0. Preliminaries

Our notation is standard and generally follows [6]. We call a vector x *extreme* if it is an extreme point of the ball of radius $\|x\|$. The consequence that 0 becomes an extreme point is unimportant.

We need the concept of *local uniform convexity* (LUR) [4, p. 145]. For x in the unit sphere of $\|\cdot\|$ and $\varepsilon < 0$, define

$$\delta(x,\varepsilon) = \inf\{1 - \|(x+y)/2\| : \|y\| = 1, \|x-y\| \geqq \varepsilon\}, \quad \text{and}$$
$$\lambda(x,\varepsilon) = \sup\{\|(x+y)/2\| : \|y\| = 1, \|x-y\| \geqq \varepsilon\}.$$

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We will say $\|\cdot\|$ is LUR at x, if for each $0<\varepsilon \leqq 2$, $\delta(x,\varepsilon)>0$ or equivalently $\lambda(x,\varepsilon)<1$. A norm is LUR if for each unit vector x it is LUR at x. It is known [4, p. 160] that each separable (even WCG) space has an equivalent LUR norm. For those that find Day [4] terse an alternate reference is Diestel [5].

§1. Pimples

Webster's seventh new collegiate dictionary defines a pimple as either a small inflamed elevation of the skin or a swelling or protuberance like a pimple. Our definition of a pimple is a new equivalent norm whose unit ball is the old unit ball with two small cones added. The new unit ball can be thought of as the convexification of the union of the old unit ball with the line segment between x and $-x$, where x isn't in the old ball. The important parameter of a pimple is the possible lengths of maximal line segments in the unit sphere with one endpoint at x.

More formally, if $\|\cdot\|$ is a norm on X, $x_0 \in X$ with $\|x_0\|=1$ and for $0<\lambda<1$, then define for $y \in X$

$$[\![y]\!] = \begin{cases} |a|\lambda & \text{if } y = ax_0, \\ \|y\| & \text{otherwise.} \end{cases}$$

Of course $[\![\cdot]\!]$ isn't a norm, we convexify it in the usual manner; namely,

$$\|y\|_\lambda = \inf\left\{\sum [\![y_i]\!] : y = \sum y_i\right\}.$$

For $0<\lambda<1$, we will call $\|\cdot\|_\lambda$ the *λ-pimple* at x_0. Let $x_\lambda = \lambda^{-1}x_0$.

The unit sphere of $\|\cdot\|_\lambda$, $S=\{y : \|y\|_\lambda = 1\}$, will contain line segments, that is, there are points $y, z \in S$, so that $\|ty+(1-t)z\|_\lambda = 1$ for each $0 \leqq t \leqq 1$. The points y and z are said to be *endpoints of a maximal line segment in the unit sphere* of $\|\cdot\|_\lambda$, if

$$L = \{ty + (1-t)z : 0 \leqq t \leqq 1\} \subset S$$

and if $L \subset L' = \{su + (1-s)v : 0 \leqq t \leqq 1\} \subset S$ then $L = L'$. Roughly speaking the new "cones", $\{x : \|x\|_\lambda = 1 < \|x\|\}$, are unions of maximal line segments in S with one endpoint at x_λ or $-x_\lambda$.

PROPOSITION. *Let $(X, \|\cdot\|)$ be a Banach space and let $\|x_0\|=1$ so that*

(1) *$\|\cdot\|$ is LUR at x_0, and*

(2) *there is $\varepsilon>0$ so that if $\|y\|=1$ and $\|x_0 - y\|<\varepsilon$, then y is an extreme point.*

Then given δ, $B>0$ and $0<m<1$, there is a λ_0 with $0<\lambda_0<1$ so that if $\lambda_0 \leqq \lambda < 1$ and $\|\cdot\|_\lambda$ is the λ-pimple at x_0, then

(3) $m\|\cdot\| \leqq \|\cdot\|_\lambda \leqq \|\cdot\|$,

(4) *if* $1=\|y\|>\|y\|_\lambda$, *then one of* $\|\pm x_0-y\|<\delta$,

(5) $x_\lambda=\lambda^{-1}x_0$ *is the only isolated extreme point of* $\|\cdot\|_\lambda$ *which satisfies* $\|x/\|x\|-x_0\|<\varepsilon$,

(6) *if w is a vector so that x_λ and $x_\lambda+w$ are endpoints of a maximal line segment in the unit sphere of $\|\cdot\|_\lambda$, then $B \geqq \|w\| \geqq \lambda^{-1}-1$.*

PROOF. Since $\lambda\|y\| \leqq [\![\lambda]\!] \leqq \|y\|$, condition (3) is true whenever $m \leqq \lambda_0$. The rest of the conclusions are not as easy. We start by observing

(A) $\|y\|_\lambda=\min\{\|y-ax_0\|+|a|\lambda : a \in R\}$.

Since the right hand side of (A) is a continuous function of a, it suffices to show (A) with "min" replaced by "inf". Now for $a \in R$, $\|y\|_\lambda \leqq [\![y-ax_0]\!]+[\![ax_0]\!] \leqq \|y-ax_0\|+|a|\lambda$, and we need only check that the left hand side of (A) is greater than or equal to the right hand side. So if $y=\Sigma y_i$, then $y=\Sigma a_jx_0+\Sigma z_k$ where we have divided the y_i's into those in span (x_0) and all others. Since

$$\sum [\![a_jx_0]\!]=\sum |a_j|[\![x_0]\!] \geqq \left|\sum a_j\right|[\![x_0]\!]=\left|\sum a_j\right|\lambda \text{ and } \sum[\![z_k]\!]=\sum\|z_k\| \geqq \left\|\sum z_k\right\|,$$

the proof of (A) is complete.

Next, for each $y \neq \pm x_\lambda$ with $1=\|y\|_\lambda<\|y\|$ we pick z_y so that $\|z_y\|_\lambda=\|z_y\|=1$ and

(B) $y=(1-|a|\lambda)z_y+|a|\lambda(\pm x_\lambda)$

(the choice of $\pm x_\lambda$ depending on if $a>0$ or $a<0$). Indeed, from (A) we obtain an a so that

(C) $1=\|y-ax_0\|+|a|\lambda$.

Now $a \neq 0$ since $1<\|y\|$ and if $\|y-ax_0\|=0$, then $y=\pm x_\lambda$. Otherwise, let $z_y=(y-ax_0)/\|y-ax_0\|$. Since (3) implies $\|y-ax_0\|_\lambda \leqq \|y-ax_0\|$, the min in (A) requires $\|z_y\|_\lambda=\|z_y\|=1$. Finally equation (B) follows as a rewrite of the equation $y=y-ax_0+ax_0$, since (C) implies $\|y-ax_0\|=1-|a|\lambda$. Clearly (B) states that y isn't an extreme point. Also it follows that x_λ is isolated from any other extreme point of $\|\cdot\|_\lambda$. Later we will show x_λ is itself extreme.

For the rest of the proof we will assume that $a>0$. (If $a<0$ we can replace x_λ by $-x_\lambda$.) Consider the line segment with endpoints z_y and x_λ. The triangle inequality implies $\|tz_y+(1-t)x_\lambda\|_\lambda \leqq 1$ for $0 \leqq t \leqq 1$. Since y is another point on this line segment with $\|y\|_\lambda=1$ and $x_\lambda \neq y \neq z_y$, we actually have

(D) $\|tz_y+(1-t)x_\lambda\|_\lambda=1$ for $0 \leqq t \leqq 1$.

Indeed if $\|w\|_\lambda < 1$ for some point w on this line segment, then for either $u = z_y$ or x_λ we can write $y = sw + (1-s)u$ and hence

$$\|y\|_\lambda \leqq s\|w\|_\lambda + (1-s)\|u\|_\lambda < s + (1-s) = 1.$$

This contradition shows (D) is true.

We need some estimates. If $\|y\| = 1$ and $\|y - x_0\| < 1 - \lambda$, then $\|y\|_\lambda \leqq \|y - x_0\| + \lambda < 1 - \lambda + \lambda = 1$ and hence $\|y\|_\lambda < \|y\|$. In particular, $\|z_y - x_0\| \geqq 1 - \lambda$. Also since $\|x_\lambda\| = \lambda^{-1}$ and $x_\lambda = x_\lambda - z_y + z_y$ we have $\|z_y - x_\lambda\| \geqq \|x_\lambda\| - \|z_y\|$ and

(E) $\|z_y - x_\lambda\| \geqq \lambda^{-1} - 1$.

If $\|y\|_\lambda = 1 < \|y\|$, then

(F) $\|y/\|y\| - x_0\| \leqq \max\{\|z_y - x_0\|, \lambda^{-1} - 1\}$.

Indeed $y/\|y\|$ is inside the triangle with vertices x_0, x_λ and z_y, hence

$$y/\|y\| - x_0 = s(z_y - x_0) + t(x_\lambda - x_0), \qquad \text{for some } s, t \geqq 0 \quad \text{with } s + t \leqq 1.$$

Also

(G) $\|z_y - x_\lambda\| \leqq \|z_y - x_0\| + \lambda^{-1} - 1$.

Consider $t = (1+\lambda)^{-1}$ and let $w = tz_y + tx_0 = tz_y + (1-t)x_\lambda$. We have $\|w\|_\lambda = 1$ so that $1 \leqq \|w\| \leqq 2t \leqq \lambda^{-1}$ and hence

(H) $(1+\lambda)/2 \leqq \|(z_y + x_0)/2\| = \|w/2t\| \leqq (1+\lambda)/2\lambda$.

We have already required $m \leqq \lambda_0 < 1$. So let $\varepsilon > 0$ be as in (2), $\delta > 0$ as in (4) and let $\xi = \min(\varepsilon, \delta, B/2)$.

Since by (1), $\|\cdot\|$ is LUR at x_0, let $\lambda(x_0, \varepsilon) < 1$ be as given in Section 0. If λ_0 is close enough to one so that

(J) $\lambda_0^{-1} - 1 < \xi$ and

(K) $(1+\lambda_0)/2 > \lambda(x_0, \xi)$,

then for $1 > \lambda \geqq \lambda_0$ and any y with $\|y\| = 1$ and $\|y - x_0\| \geqq \xi$ we have $\|(y + x_0)/2\| < (1+\lambda)/2$. Since by (H), z_y fails this inequality, it follows that

(L) $\xi > \|x_0 - z_y\|$.

Thus by (E) and (F), if $\|y\|_\lambda \neq \|y\|$, then $\|y/\|y\| - x_0\| < \delta, \varepsilon$, which proves part (4) of the proposition. Furthermore, using (E), (G), (L) and then (J) we have

(M) $\lambda^{-1} - 1 \leqq \|z_y - x_\lambda\| \leqq \xi + \lambda^{-1} - 1 < 2\xi < B$.

This shows the estimate in (6) is satisfied with $w = z_y - x_\lambda$. Later we will show these are the only w's with x_λ and $x_\lambda + w$ endpoints of a maximal line segment in the unit sphere of $\|\cdot\|_\lambda$.

Now for extreme points. We have shown x_λ is isolated from the other extreme points of $\|\cdot\|_\lambda$. To see x_λ itself is extreme, suppose it isn't. Then $x_\lambda = (u+v)/2$, $u \neq v$ and $\|u\|_\lambda = \|v\|_\lambda = 1$. Extend the function $y \to z_y$ defined for $1 = \|y\|_\lambda <$

$\|y\|$ and $y \neq \pm x_\lambda$, by letting $z_y = y$ if $\|y\| = \|y\|_\lambda = 1$. It follows that x_λ is a convex combination of z_u and z_v. But this is impossible since $\|x_\lambda\| = \lambda^{-1} > 1$ and both $\|z_u\| = \|z_v\| = 1$. Thus x_λ is an isolated extreme point.

Next suppose $\|y\| = \|y\|_\lambda = 1$ and $\|y - x_0\| < \varepsilon$ so that y is an extreme point $\|\cdot\|$ by (2). We will show y is an extreme point of $\|\cdot\|_\lambda$. Suppose y isn't an extreme of $\|\cdot\|_\lambda$, so there are $u \neq v$, $\|u\|_\lambda = \|v\|_\lambda = 1$ with $y = (u+v)/2$. We can't have both $\|u\| = 1 = \|v\|$ since y is extreme in $\|\cdot\|$. Now by (B), y is a convex combination of z_u, z_v and x_λ. In particular, there is a $w \neq y$ which is a convex combination of z_u and z_v and so that y is a convex combination of w and x_λ.

Now $\|w\|_\lambda \leqq \|w\| \leqq 1$, since w is a convex combination of z_u and z_v. Since $\|y\|_\lambda = 1$ and y is a convex combination of w and x_λ, it follows that $\|w\|_\lambda = \|w\| = 1$. Now for similar reasons for each $0 \leqq t \leqq 1$, $x(t) = ty + (1-t)w$ also satisfies $\|x(t)\|_\lambda = \|x(t)\| = 1$. But this implies that $x(t)$ is not an extreme point of $\|\cdot\|$ for $t > 0$ as required by (2).

In summary, we have shown that if $\|y\| = 1$ and $\|x_0 - y\| < \varepsilon$ then $y/\|y\|$ is an extreme point of $\|\cdot\|_\lambda$ if and only if $y = x_\lambda$ or $\|y\|_\lambda = \|y\|$. It follows that $y(t) = (ty + (1-t)x_\lambda)/\|ty + (1-t)x_\lambda\|$ is an extreme point of both $\|\cdot\|$ and $\|\cdot\|_\lambda$ for $t \geqq 1$ (and t near enough to one) whenever $\|y\| = \|y\|_\lambda$ and $\|x_0 - y\| < \varepsilon$. It follows that a x_λ is the only isolated extreme point in the neighborhood required by (5).

In particular, since the vector z_y from (B) is in this neighborhood by (L), its choice is unique. That is, there is exactly one z_y satisfying (B). Let x_λ and $x_\lambda + w$ be the endpoints of a maximal line segment in the unit sphere of $\|\cdot\|_\lambda$. Let $0 < t \leqq 1$ be near enough zero so that $y = t(x_\lambda + w) + (1-t)x_\lambda$ satisfies $\|y\| \neq \|y\|_\lambda$. By the uniqueness of z_y, we have $z_y = x_\lambda + w$. Therefore, by (M), the estimate in (6) is true. □

REMARKS. (1) If $\|\cdot\|$ is LUR at y_0 with $\|y_0\| = 1$ and $\|x_0 - y_0\| > \delta$, then $\|\cdot\|_\lambda$ is LUR at y_0, because LUR depends only on vectors near y_0.

(2) If $\|y_0\| = 1$ and $\|x_0 - y_0\| > \delta$, then we can put a pimple at y_0 on $\|\cdot\|_\lambda$ whose unit ball is the union of the pimplies at x_0 on $\|\cdot\|$ and y_0 on $\|\cdot\|$. That is, the "cones" for the two pimples are not visible to each other.

(3) Something like LUR at x_0 is required to obtain the upper bound on $\|z_y - x_\lambda\|$. Indeed, if $x_0 = e_1$ of the usual basis for l_1, then $z_y \in [e_i]_2^\infty$.

§2. Infinite acne

THEOREM 1. *Each separable Banach space can be given an equivalent norm in which the only isometries are ± identity.*

PROOF. Let X be a separable Banach space and let $(y_n)_0^\infty$ be a linearly independent set with dense linear span. We assume that the norm on X is locally uniformly convex, which we may by Section 0. The first step is to inductively pick a sequence $(x_n)_0^\infty$ so that (x_n) has dense linear span and

(A) for $n > 0$, $\|x_0 - x_n\| \leqq \|x_0 + x_n\|/2$ and

(B) for $m < n$, $\|x_m - x_n\| \geqq \frac{1}{3}$,

(C) $\mathrm{span}(x_i)_{i=0}^n = \mathrm{span}(y_i)_{i=0}^n$.

Note that (C) will imply that (x_i) has dense linear span.

Start the induction by letting $x_0 = y_0/\|y_0\|$. Suppose $x_0, \dots, x_{n-1}$ have been chosen. Let B be the span of these vectors and E be the span of B and y_n. Pick $z \in E$ so that $\|z\| = 1$ and $\mathrm{dist}(z, B) = 1$. Note that $\|x_0 - z\| \leqq 2$ and pick $x_n = ax_0 + z/3$ so that $\|x_n\| = 1$ and $a > 0$. Convexity implies $a \geqq \frac{2}{3}$. Also, $1 = \|ax_0 + z/3\| \geqq |a| - \frac{1}{3}$ implies $a \leqq \frac{4}{3}$. Hence $\|x_0 - x_n\| \leqq \frac{2}{3}$ and $\|x_0 + x_n\| \geqq \frac{4}{3}$. Therefore $\|x_0 - x_n\| \leqq \|x_0 + x_n\|/2$. The $z/3$ term in x_n implies that for $m < n$, $\|x_m - x_n\| \geqq \frac{1}{3}$.

The next step is to construct pimples λ_n at x_n inductively so that if $\|\cdot\|_n$ is the pimple norm then:

(D) $\frac{8}{9}\|\cdot\| \leqq \|\cdot\|_n$ (i.e., $\frac{8}{9} \leqq \lambda_n$),

(E) if $1 = \|y\| > \|y\|_n$, then either $\|y - x_n\|$ or $\|y + x_n\|$ is less than $\frac{1}{9}$,

(F) for every n there exist numbers $a_n, b_n > 0$ so that if w is a vector so that $\lambda_n^{-1}x_n$ and $\lambda_n^{-1}x_n + w$ is a maximal line segment in the pimple then $a_n \geqq \|w\| \geqq b_n$, and

(G) $8b_n/9 > a_{n+1}$.

All these conditions can be satisfied by making $\lambda_n < 1$ near enough to one by the Proposition.

Now let $|||\cdot|||$ be the equivalent norm obtained by adding the above pimples to the unit ball of $\|\cdot\|$ (that is, $|||x||| = \min\|x\|_n$). We have $\frac{8}{9}\|\cdot\| \leqq |||\cdot||| \leqq \|\cdot\|$. Now we will show that $(X, |||\cdot|||)$ has only trivial isometries. Let T be an isometry for $|||\cdot|||$. Since $E = (\pm\lambda_n^{-1}x_n)$ is the set of isolated extreme points of $|||\cdot|||$, T maps E onto itself. If $n < m$, then T cannot map $\lambda_n^{-1}x_n$ to either $\pm\lambda_m^{-1}x_m$. Since if w (respectively, w') is a vector so that $\lambda_n^{-1}x_n$ and $\lambda_n^{-1}x_n + w$ (respectively, $\lambda_m^{-1}x_m$ and $\lambda_m^{-1}x_m + w'$) are endpoints of a maximal line segment in the unit sphere of $|||\cdot|||$, then

$$|||w||| \geqq \tfrac{8}{9}\|w\| \geqq 8b_n/9 > a_m \geqq \|w'\| \geqq |||w'|||.$$

Thus T maps $\lambda_n^{-1}x_n$ into $\pm\lambda_n^{-1}x_n$, and hence $Tx_n = \pm x_n$. Replacing T by $-T$ if necessary, we may assume $Tx_0 = x_0$. If $Tx_n = -x_n$, then

$$|||T(x_0 + x_n)||| = |||x_0 - x_n||| \leqq \|x_0 - x_n\| \leqq \|x_0 + x_n\|/2 \leqq \tfrac{9}{16}|||x_0 + x_n|||$$

contradicting the fact T is an isometry. Therefore T is the identity on the closed linear span of $(x_n) = X$. □

REMARK. The equivalent norm $|||\cdot|||$ can be made as near to LUR norm $\|\cdot\|$ as we like. Thus equivalent norms with trivial isometries are dense in the collection of equivalent norms since the equivalent LUR norms are dense. The standard construction of an equivalent LUR norm can be combined with Asplund averaging to show the LUR norms are dense. Both of these topics are in [4] and [5].

§3. Uncountable acne

THEOREM 2. *The non-separable Hilbert space $l_2(\mathbf{R})$ has an equivalent norm with only trivial isometries.*

PROOF. Let $\|\cdot\|$ be the usual norm on $l_2(\mathbf{R})$ and let $\{e_\alpha: \alpha \in [30,31]\}$ be an orthonormal basis for this space. For $\alpha \in [30,31]$ we define

$$\begin{aligned}
\lambda_\alpha &= \cos \alpha^\circ, \\
\mu_\alpha &= \sec \alpha^\circ = \lambda_\alpha^{-1}, \\
x_\alpha &= \begin{cases} e_{30} & \text{if } \alpha = 30, \\ \cos 89^\circ e_{30} + \cos 1^\circ e_\alpha & \text{if } \alpha \neq 30, \end{cases} \\
y_\alpha &= \mu_\alpha x_\alpha .
\end{aligned}$$

Consider the λ_α-pimple at x_α (see Fig. 1). Observe that if $\|z\|$ is different from the pimple norm of z, then the angle between z and one of $\pm x_\alpha$ is less than $\alpha^\circ \leqq 31^\circ$. Also note that the vector w is so that y_α and $y_\alpha + w$ are the endpoints of a maximal line segment in the unit sphere of this pimple, if and only if $\|w\| = \tan \alpha^\circ$ and the angle between w and x_α is $90^\circ - \alpha^\circ$.

Let $|||\cdot|||$ be the equivalent norm obtained by adding all the pimples above to the unit ball of $\|\cdot\|$. If w is so that y_α and $y_\alpha + w$ are the endpoints of a maximal line segment in the unit sphere of $|||\cdot|||$, then $|||w||| \leqq \|w\| = \tan \alpha^\circ$. If $\beta \neq \gamma$ are elements in $(30,31]\backslash\{\alpha\}$ then

$$w = \tan \alpha^\circ(\sin \alpha^\circ x_\alpha + \cos \alpha^\circ(e_\beta + e_\gamma)/\sqrt{2})$$

is such a vector, and $|||w||| = \|w\| = \tan \alpha^\circ$, since the angle between w and any $\pm x_\alpha$ is greater than 31°. Similarly,

$$\begin{aligned}
|||y_{30} - y_\alpha|||^2 = \|y_{30} - y_\alpha\|^2 &= \mu_{30}^2 + \mu_\alpha^2 - 2\mu_{30}\mu_\alpha \cos 89^\circ \\
\neq \mu_{30}^2 + \mu_\alpha^2 + 2\mu_{30}\mu_\alpha \cos 89^\circ &= \|y_{30} + y_\alpha\|^2 = |||y_{30} + y_\alpha|||^2.
\end{aligned}$$

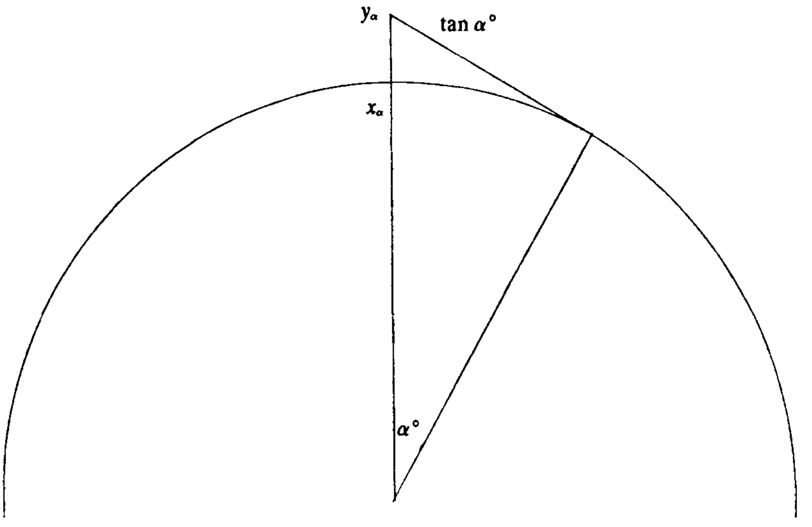

Fig. 1.

Now let T be an isometry for $|||\cdot|||$. Since $E = \{\pm y_\alpha : \alpha \in [30,31]\}$ is the collection of the isolated extreme points of $|||\cdot|||$, T maps E onto itself. If $30 \leqq \alpha < \beta \leqq 31$ then $Ty_\beta \neq \pm y_\alpha$ since there is a line segment with endpoint y_β of length $\tan \beta°$ in the unit sphere of $|||\cdot|||$, but all those with endpoint y_α have length $\leqq \tan \alpha° < \tan \beta°$. Hence $Ty_\alpha = \pm y_\alpha$. Replacing T with $-T$ if necessary, we may assume $Ty_{30} = y_{30}$. Now for all $\alpha \in [30,31]$, $Ty_\alpha \neq -y_\alpha$, since $|||\, y_{30} + y_\alpha \,||| \neq |||\, y_{30} - y_\alpha \,|||$. Thus T is the identity on E and hence on $l_2(\mathbf{R})$ its closed linear span. $\square$

REMARKS. (1) The "same" construction yields equivalent norms with trivial isometries for Hilbert spaces with dimension d and $2 \leqq d \leqq c = \mathrm{card}(\mathbf{R})$.

(2) If we had centered the pimples at e_α rather than x_α, then the group of isometries would be $\{-1,1\}^{\mathbf{R}}$.

REFERENCES

1. S. F. Bellenot, *Isometries of James space*, to appear.

2. A. Brunel and L. Sucheston, *On B-convex spaces*, Math. Systems Theory **7** (1973), 294–299.

3. W. J. Davis, *Separable Banach spaces with only trivial isometries*, Rev. Roum. Math. Pures Appl. **16** (1971), 1051–1054.

4. M. M. Day, *Normed Linear Spaces*, 3rd ed., Springer-Verlag, New York/Berlin, 1973.

5. J. Diestel, *Geometry of Banach Spaces — Selected Topics*, Lecture Notes in Math., Vol. 485, Springer-Verlag, New York/Berlin, 1975.

6. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, Sequence Spaces*, Springer-Verlag, New York/Berlin, 1977.